

UNIFORM CENTRAL LIMIT THEOREM FOR MARTINGALES.

L.Sirota

Department of Mathematics and computer science.
Bar-Ilan University, 84105, Ramat Gan, Israel.

E-mail: sirota3@bezeqint.net

ABSTRACT.

We study some sufficient conditions imposed on the sequence of martingale differences (m.d.) in the separable Banach spaces of continuous functions defined on the metric compact set for the Central Limit Theorem in this space.

We taking into account the classical entropy terms, and use the theory of the so - called Grand Lebesgue Spaces of random variables having power and exponential decreasing tail of distribution.

Key words and phrases: Central Limit Theorem (CLT) in Banach space, tail and tail function, space of continuous function, upper and lower estimates, natural function, embedding, moments, filtration, martingale and martingale differences (m.d.), random variable or random vector (r.v.), distribution, weak convergence, entropy and entropy integral, compact metric space, covering numbers and integral, natural distance, ball, covariation function, Grand Lebesgue Spaces (GLS).

2000 Mathematics Subject Classification. Primary 37B30, 33K55; Secondary 34A34, 65M20, 42B25.

1 Notations. Statement of problem.

Let

A. B be Separable Banach Space with a norm $\|\cdot\|_B$, briefly: $B = (B, \|\cdot\|_B)$, equipped with Borelian sigma-algebra;

B. (Ω, F, P) be probability triple with expectation E ;

C. $F(i)$, $i = 0, 1, 2, \dots$ be filtration, i.e. monotonically non-decreasing flow (sequence) of sigma-subfields of source sigma-field F such that $F(0)$ is trivial sigma-algebra: $F(0) = \{\emptyset, \Omega\}$;

D. A sequence of *centered* martingale differences (m.d.) $\xi(i)$, $i = 1, 2, \dots$, i.e. Borelian distributed r.v. with values in the space B such that

$$\mathbf{E}\xi(i)/F(k) = 0, \quad k < i; \quad \mathbf{E}\xi(i)/F(i) = \xi(i) \pmod{\mathbf{P}}. \quad (1.0)$$

The last equalities (1.0) may be explained as follows. For all non-random elements b^* from dual (conjugate) space B^*

$$\mathbf{E} b^*(\xi(i))/F(k) = 0, \quad k < i; \quad \mathbf{E} b^*(\xi(i))/F(i) = b^*(\xi(i)) \pmod{\mathbf{P}}. \quad (1.0a)$$

Denote

$$S(n) = \sum_{i=1}^n \xi(i);$$

then $(S(n), F(n))$ is really B space valued mean zero martingale.

Definition 1.1. We will say that the martingale $(S(n), F(n))$ or simple $S(n)$ satisfies the Central Limit Theorem (CLT) in the Banach space B , if the sequence of distributions of a random variables

$$\eta(n) := n^{-1/2} S(n) = n^{-1/2} \sum_{i=1}^n \xi(i) \quad (1.1)$$

i.e. under classical norming sequence $1/\sqrt{n}$, converges weakly as $n \rightarrow \infty$ to the non-zero Gaussian distributed r.v. $\eta(\infty)$ in the space B .

Evidently, $\mathbf{E}\eta(\infty) = 0$. The covariation operator $R_{\eta(\infty)}(\cdot)$ of the limiting r.v. $\eta(\infty)$ has a form

$$R_{\eta(\infty)}(b^*) \stackrel{def}{=} \mathbf{E}(b^*(\eta(\infty)))^2 = \lim_{n \rightarrow \infty} n^{-1} \sum_{i=1}^n R_{\eta(i)}(b^*). \quad (1.2)$$

We will suppose that the Gaussian r.v. $\eta(\infty)$ there exists, belongs to the space $B \pmod{\mathbf{P}}$ and that for all $b^ \in B^*$ the one-dimensional r.v. $b^*(\eta(n))$ converge in distribution as $n \rightarrow \infty$ to one for the r.v. $b^*(\eta(\infty))$.*

It remains to establish only the weak compactness, i.e. in Yu.V.Prokhorov's sense [29] of distributions of the r.v. $\eta(n)$ in the space B to deduce the CLT for the martingale $S(n)$ in this space.

Note that the one-dimensional CLT for martingales is described in the famous book [31]; see also [1], [33], [34], [39] etc. The CLT in Banach spaces for independent variables is considered in monographs [4], [15], [18] and in many articles.

2 Uniform CLT for martingales in entropy terms.

This section may be considered as a simplification of the article [1].

Let $(X = \{x\}, d)$ be compact metric space relative some distance (or semi-distance) $d = d(x_1, x_2)$, and let $\{\xi_i\} = \{\xi_i(x)\}$, $x \in X$ be centered martingale differences relative the index i random processes (r.p.) (fields, r.f.); the continuity with probability one of ones it follows from conditions of a next theorem.

Denote as ordinary

$$\eta_n(x) := n^{-1/2} \sum_{i=1}^n \xi_i(x), \quad (2.0)$$

$$\psi(p) := \sup_i \sup_{x \in X} |\xi_i(x)|_p = \sup_i \sup_{x \in X} [\mathbf{E}|\xi_i(x)|^p]^{1/p}. \quad (2.1)$$

We suppose in what follows that the introduced function $\psi = \psi(p)$ is finite at last for some value B , $B = \text{const} > 2$; may be $B = \infty$; then evidently it is finite for all the values p from the set $[1, B)$.

The function $\psi(\cdot)$ is called in the theory of Grand Lebesgue Spaces (GLS) as a *natural function* for the family of the random variables $\{\xi_i(x)\}$, $i = 1, 2, \dots$; $x \in X$.

We recall here briefly the definition and some simple properties of the so-called Grand Lebesgue spaces; more detail investigation of these spaces see in [8], [12], [14], [16], [18], [19]; see also reference therein.

Recently appear the so-called Grand Lebesgue Spaces $GLS = G(\psi) = G\psi = G(\psi; A, B)$, $A, B = \text{const}$, $A \geq 1$, $A < B \leq \infty$, spaces consisting on all the random variables (measurable functions) $f : \Omega \rightarrow R$ with finite norms

$$\|f\|G(\psi) \stackrel{\text{def}}{=} \sup_{p \in (A, B)} [\|f\|_p / \psi(p)]. \quad (2.2)$$

Here $\psi(\cdot)$ is some continuous positive on the *open* interval (A, B) function such that

$$\inf_{p \in (A, B)} \psi(p) > 0, \quad \psi(p) = \infty, \quad p \notin (A, B).$$

We can accept in the sequel $A = \text{const} = 2$, and following $B > 2$, taking into account the application in the theory of CLT in the space of all numerical (real or complex) continuous functions $C(X)$.

We will denote

$$\text{supp}(\psi) \stackrel{\text{def}}{=} (A, B) = \{p : \psi(p) < \infty, \}$$

The set of all ψ functions with support $\text{supp}(\psi) = (A, B)$ will be denoted by $\Psi(A, B)$.

This spaces are rearrangement invariant, see [3], and are used, for example, in the theory of probability [14], [18], [19]; theory of Partial Differential Equations [8], [12]; functional analysis [8], [12], [16], [19]; theory of Fourier series, theory of martingales, mathematical statistics, theory of approximation etc.

The function $\psi(\cdot)$ introduced in (2.1) generated the bounded *natural distances* $d_i = d_i(x_1, x_2)$ and $\bar{d} = \bar{d}(x_1, x_2)$, $x_1, x_2 \in X$ (more exactly, semi-distances) on the set X :

$$d_i(x_1, x_2) \stackrel{def}{=} \|\xi_i(x_1) - \xi_i(x_2)\|G\psi, \quad (2.3)$$

so that

$$d_i(x_1, x_2) \leq 2; \quad \forall i \Rightarrow \|\xi_i(x_1) - \xi_i(x_2)\|G\psi \leq d_i(x_1, x_2);$$

$$\bar{d}(x_1, x_2) \stackrel{def}{=} \sup_n \left[\sqrt{n^{-1} \sum_{i=1}^n d_i^2(x_1, x_2)} \right]. \quad (2.3a)$$

Let us introduce for any subset V , $V \subset X$ and for arbitrary semi - distance on the set X $d = d(x_1, x_2)$, $x_1, x_2 \in X$ the so-called *entropy* $H(V, d, \epsilon) = H(V, \epsilon)$ as a natural logarithm of a minimal quantity $N(V, d, \epsilon) = N(V, \epsilon) = N$ of a balls $S(V, t, \epsilon)$, $t \in V$:

$$S(V, t, \epsilon) \stackrel{def}{=} \{s, s \in V, d(s, t) \leq \epsilon\},$$

which cover the set V (covering numbers):

$$N = N(V, d, \epsilon) = \min\{M : \exists\{t_i\}, i = 1, 2, \dots, M, t_i \in V, V \subset \cup_{i=1}^M S(V, t_i, \epsilon)\}, \quad (2.4)$$

and we denote also $D = D(d) = \text{diam}(X, d) = \sup_{x_1, x_2 \in X} d(x_1, x_2)$,

$$H(V, d, \epsilon) = \log N; \quad S(t_0, \epsilon) \stackrel{def}{=} S(X, t_0, \epsilon), \quad H(d, \epsilon) \stackrel{def}{=} H(X, d, \epsilon). \quad (2.4a)$$

It follows from Hausdorff's theorem that $\forall \epsilon > 0 \Rightarrow H(V, d, \epsilon) < \infty$ iff the metric space (V, d) is precompact set, i.e. is the bounded set with compact closure.

We will distinguish in the sequel two cases: $B < \infty$, *finite case*, and $B = \infty$, *infinite case*.

Finite case.

The probabilistic Grand Lebesgue Spaces $G\Psi(2, B)$ with $2 < B < \infty$ are in detail investigated in articles [16], [25], including consideration many examples.

We will use in this case the following inequality for the arbitrary sequence $\{\zeta_k\}$ of martingale difference, see the famous article of A.Osekovski [17], see also [32]:

$$\left| n^{-1/2} \sum_{k=1}^n \zeta_k \right|_p \leq K_{Os} \cdot \frac{p}{\ln p} \cdot \sqrt{\sum_{k=1}^n |\zeta_k|_p^2 / n}, \quad (2.5)$$

where the "Osekowski's" constant K_{Os} is less than 15.5879.

It is interest to note that at the same estimate was before obtained by H.Rosenthal [30] for independent variables; in this case the exact value of this constant ("Rosenthal's constant") $C_R \approx 0.6535$, see [26].

Infinite case.

The spaces $G\Psi(2, \infty)$ are convenient, e.g., for the investigation of the random variables and vectors with exponential decreasing tail of distribution. Indeed, if for some non-zero r.v. ξ we have $0 < \|\xi\|G(\psi) < \infty$, then for all positive values u

$$\mathbf{P}(|\xi| > u) \leq 2 \exp\left(-\overline{\psi}^*(\log x / \|\xi\|G(\psi))\right), \quad (2.6)$$

where $\overline{\psi}(p) = p \log \psi(p)$ and the symbol g^* denotes some modification of the Young-Fenchel, or Legendre transform of the function g :

$$g^*(y) = \sup_{x \geq 2} (xy - g(x)).$$

see [14], [18], chapters 1,2.

As a consequence: if

$$\forall x > e^2 \Rightarrow \overline{\psi}^*(\log x) > 0,$$

then the space $G\psi$ coincides with *exponential* Orlicz's space over our probabilistic space (Ω, F, \mathbf{P}) with N -function of a form

$$N(u) = \exp(\overline{\psi}^*(\log |u|)), \quad |u| > e^2; \quad N(u) = C \cdot u^2, \quad |u| \leq e^2. \quad (2.7)$$

Conversely: if a r.v. ξ satisfies (2.6), then $\xi \in G\psi$, $\|\xi\|G(\psi) < \infty$.

Example 2.1. Introduce as a particular case the following norm for the r.v. ξ :

$$K := \|\xi\|_{(q)} := \sup_{p \geq 2} \left[\frac{|\xi|_p}{p^{1/q}} \right], \quad q = \text{const} > 0;$$

then $K = \|\xi\|_{(q)} \in (0, \infty) \Leftrightarrow$

$$T(\xi, x) := \max(\mathbf{P}(\xi > x), \mathbf{P}(\xi < -x)) \leq \exp(-C(q)(x/K)^q), \quad x > 1. \quad (2.8)$$

So, the theory of $G\psi$ spaces of random variables gives a very convenient apparatus for investigation of a random variables with exponential decreasing tails of distribution.

Let $\psi \in \Psi(2, B)$; we introduce the so-called Rosenthal's transform $\psi_R(\cdot)$ as follows:

$$\psi_R(p) := \frac{p}{\log p} \cdot \psi(p). \quad (2.9)$$

Evidently, $\psi(p) \leq e \cdot \psi_R(p)$; and in addition if $B < \infty$, then

$$\psi_R(p) \asymp \psi(p),$$

and this is not true if $B = \infty$.

Let us denote for arbitrary function $\psi \in \Psi$

$$\psi_*(x) := \inf_{y \in (0,1)} (xy + \log \psi(1/y)),$$

and introduce the following functional ("entropy integral, covering integral")

$$J(\psi, d) \stackrel{\text{def}}{=} \int_0^D \exp(\psi_*(\log 2 + H(X, d, \epsilon))) d\epsilon. \quad (2.10)$$

Remark 2.1. If we introduce the *discontinuous* function

$$\psi_r(p) = 1, \quad p = r; \quad \psi_r(p) = \infty, \quad p \neq r, \quad p, r \in (A, B)$$

and define formally $C/\infty = 0$, $C = \text{const} \in R^1$, then the norm in the space $G(\psi_r)$ coincides with the L_r norm:

$$\|f\|_{G(\psi_r)} = |f|_r.$$

Thus, the Bilateral Grand Lebesgue spaces are direct generalization of the classical exponential Orlicz's spaces as well as of the classical Lebesgue-Riesz spaces L_r .

We need to introduce again some notations. Let $T(x)$, $x > 0$ be a *tail - function*, i.e. such that $T(0) = 1$, $T(\cdot)$ is monotonically decreasing, right continuous and such that $T(\infty) = 0$.

We denote for the tail-function $T(\cdot)$ the following operator (non-linear)

$$W[T](x) = \min \left(1, \inf_{v>0} \left[\exp(-x^2/(8v^2)) - \int_v^\infty x^2 dT(x) \right] \right), \quad (2.11)$$

if there exists the second moment

$$\int_0^\infty x^2 |dT(x)| < \infty.$$

Lemma 2.1. (See [27]). Let $\xi(i)$ be a sequence of *centered* martingale-differences relative to some filtration $\{F(i)\}$ and $T(\xi(i), x) \leq T(x)$, $T(x)$ be some tail-function. Then at $x > 1$

$$\sup_{b: \sum b^2(i)=1} T \left(\sum_i b(i) \xi(i), x \right) \leq W[T](x). \quad (2.12)$$

In particular,

$$\sup_n T \left(n^{-1/2} \sum_{i=1}^n \xi(i), x \right) \leq W[T](x). \quad (2.12a)$$

If for instance

$$\exists q, K = \text{const} > 0 \Rightarrow T(x) \leq \exp(-(x/K)^q), \quad x \geq 0,$$

then

$$\sup_n T \left(n^{-1/2} \sum_{i=1}^n \xi(i), x \right) \leq \exp \left(-C(q)(x/K)^{2q/(2+q)} \right). \quad (2.12b)$$

Denote also

$$\sigma^2 = \sigma^2(\{\xi_i(\cdot)\}) = \inf_{x \in X} \sup_n \left[n^{-1} \sum_{k=1}^n \text{Var}(\xi_k(x)) \right]. \quad (2.13)$$

Theorem 2.1. ("Power" level.)

Suppose that for our sequence of functional martingale differences $\{\xi_i(\cdot)\}$

$$\sigma^2(\{\xi_i(\cdot)\}) < \infty \quad (2.14)$$

and

$$J(\psi_R, \bar{d}) = \int_0^{D(\bar{d})} \exp \left(\psi_{R,*}(\ln 2 + H(X, \bar{d}, \epsilon)) \right) d\epsilon < \infty. \quad (2.15)$$

Then the family of distributions of the sequence of random fields $\{\eta_n(\cdot)\}$ is weakly compact in the space $C(X, \bar{d})$.

Proof.

0. As long as $\psi(p) \leq e \cdot \psi_R(p)$, and

$$d_k(x_1, x_2) \leq C\bar{d}(x_1, x_2),$$

it follows from the condition (2.15) the convergence of the following entropy integral for the individual r.f.

$$J(\psi, \bar{d}) = \int_0^{D(\bar{d})} \exp \left(\psi_*(\ln 2 + H(X, \bar{d}, \epsilon)) \right) d\epsilon < \infty. \quad (2.15a)$$

Therefore, each r.f. $\xi_k(x)$ is continuous with probability one relative the distance $\bar{d}(x_1, x_2)$.

1. The condition (2.14) imply that there exists at least one point x_0 in the set X for which

$$\sup_n \left[n^{-1} \sum_{k=1}^n \text{Var}(\xi_k(x_0)) \right] < \infty.$$

Since the martingale differences are centered and non-correlated,

$$\sup_n \mathbf{E} [\eta_n(x_0)]^2 < \infty.$$

Therefore, the family of distributions on real line of one-dimensional r.v. $\{\eta_n(x_0)\}$ is weakly compact.

2. We apply the inequality (2.5) for the variables $\zeta_k = \xi_k(x_1) - \xi_k(x_2)$, where x_1, x_2 are non-random elements of the set X such that $\bar{d}(x_1, x_2) > 0$ (the case $\bar{d}(x_1, x_2) = 0$ is trivial):

$$|\eta_n(x_1) - \eta_n(x_2)|_p \leq K_{Os} \cdot \frac{p}{\ln p} \cdot \sqrt{\left[n^{-1} \sum_{k=1}^n |\xi_k(x_1) - \xi_k(x_2)|_p^2 \right]} \leq$$

$$K_{Os} \cdot \frac{p}{\ln p} \cdot \psi(p) \cdot \bar{d}(x_1, x_2) = K_{Os} \cdot \psi_R(p) \cdot \bar{d}(x_1, x_2), \quad (2.16)$$

or equally

$$\sup_n ||\eta_n(x_1) - \eta_n(x_2)||_{G\psi_R} \leq K_{Os} \cdot \bar{d}(x_1, x_2). \quad (2.16a)$$

The statement of theorem 2.1 it follows from theorem 4.4.2 of the monograph [18], chapter 4, section 4.

Example 2.1. Let $\psi(p) = \psi_r(p)$ for some $r = \text{const} \geq 2$; then the space $G\psi_r$ coincides with the classical Lebesgue - Riesz space $L_r = L_r(\Omega, \mathbf{P})$.

Then the condition (2.15) is equal to the following famous G.Pizier's condition

$$\int_0^{D(d_r)} N^{1/r}(X, d_r, \epsilon) d\epsilon < \infty,$$

i.e. as in the independent case, see e.g. [34]. Here $d_r(\cdot, \cdot)$ is the so-called natural Pizier distance

$$d_r(x_1, x_2) = \sup_k |\xi_k(x_1) - \xi_k(x_2)|_r.$$

Let us consider now the so-called "exponential level". Let again $q = \text{const} > 0$; define the following distance on the set X

$$\rho_q(x_1, x_2) := \sup_k ||\xi_k(x_1) - \xi_k(x_2)||_{(q)}. \quad (2.17)$$

It will be presumed the boundedness of this distance, as well as the finiteness of the variable $\sigma^2(\{\xi_i(\cdot)\})$.

Theorem 2.2. Suppose in addition

$$\int_0^{D(\rho_q)} H^{(2+q)/(2q)}(X, \rho_q, \epsilon) d\epsilon < \infty. \quad (2.18)$$

Then the family of distributions of the sequence of random fields $\{\eta_n(\cdot)\}$ is weakly compact in the space $C(X, \rho_q)$.

Proof. We apply the inequality (2.18) and described below the properties of Grand Lebesgue Spaces:

$$\sup_n ||\eta_n(x_1) - \eta_n(x_2)||_{(2q/(q+2))} \leq C_1(q) \rho_q(x_1, x_2). \quad (2.19)$$

The proposition of theorem 2.2 follows immediately from theorem 4.3.2 of the book [18], chapter 4, section 4.3; also [14].

Remark 2.1. In the *independent case*, i.e. when the r.f. $\xi_k(x)$ are in addition commonly independent, the condition (2.18) looks as follows:

$$\int_0^{D(\rho_q)} H^{1/q}(X, \rho_q, \epsilon) d\epsilon < \infty,$$

see [14]; i.e. is unlike in comparison to the general (martingale) case, in contradiction to the "power case".

The eventually value of power of the variable $H(X, \rho_q, \epsilon)$ in the condition (2.18) for martingales is now unknown.

Example 2.2. The condition (2.18) is satisfied if for example X is bounded closed subset of the Euclidean space R^d , $d = 1, 2, \dots$ and

$$\rho_q(x_1, x_2) \leq C_1 |x_1 - x_2|^\alpha, \quad x_1, x_2 \in X, \quad 0 < \alpha = \text{const} \leq 1.$$

In this case

$$N(X, \rho_q, \epsilon) \leq C_2(\alpha, q) \epsilon^{-d/\alpha}, \quad \epsilon > 0.$$

3 Concluding remarks.

It is interesting by our opinion to obtain the CLT for functional martingales in the space $C(X, d)$ in more modern terms of majorizing measures in the spirit of the article of B.Heinkel [11], see also [5], [6], [7], [37], [38]; as well as in the Hölder's space, see [13], [35], [36], by means of martingale inequalities [2], [27].

References

- [1] JONGSIG BAE, DOOBAE JUN, AND SHLOMO LEVENTAL. *The uniform CLT for martingale differences arrays under the uniformly integrable entropy*. Bull. Korean Math. Soc., **47**, (2010), No. 1, pp. 39-51. DOI 10.4134/BKMS.2010.47.1.039
- [2] BARLOW M.T. AND YOR M. *Semimartingale inequalities via the Garsia-Rodemich-Rumsey lemma, and applications to local times*. J. Funct. Anal.; **49**(2), (1982), 198-229.
- [3] BENNETT C. AND SHARPLEY R. *Interpolation of operators*. Orlando, Academic Press Inc., 1988.
- [4] DUDLEY R.M. *Uniform Central Limit Theorem*. Cambridge University Press, 1999.
- [5] FERNIQUE X. (1975). *Regularite des trajectoires des fonction aleatoires gaussiennes*. Ecole de Probablite de Saint-Flour, IV - 1974, Lecture Notes in Mathematic. **480**, 1-96, Springer Verlag, Berlin.

- [6] FERNIQUE X, *Caracterisation de processus de trajectoires majores ou continues*. Seminaire de Probabilits XII. Lecture Notes in Math. 649, (1978), 691706, Springer, Berlin.
- [7] FERNIQUE X. *Regularite de fonctions aleatoires non gaussiennes*. Ecole de Ete de Probabilits de Saint-Flour XI-1981. Lecture Notes in Math. 976, (1983), 174, Springer, Berlin.
- [8] FIORENZA A., AND KARADZHOV G.E. *Grand and small Lebesgue spaces and their analogs*. Consiglio Nazionale Delle Ricerche, Istituto per le Applicazioni del Calcolo Mauro Picone, Sezione di Napoli, Rapporto tecnico n. 272/03, (2005).
- [9] FROLOV A.S., TCHENTZOV N.N. *On the calculation by the Monte-Carlo method definite integrals depending on the parameters*. Journal of Computational Mathematics and Mathematical Physics, (1962), V. 2, Issue 4, p. 714-718 (in Russian).
- [10] GRIGORJEVA M.L., OSTROVSKY E.I. *Calculation of Integrals on discontinuous Functions by means of depending trials method*. Journal of Computational Mathematics and Mathematical Physics, (1996), V. 36, Issue 12, p. 28-39 (in Russian).
- [11] HEINKEL B. *Measures majorantes et le theoreme de la limite centrale dans $C(S)$* . Z. Wahrscheinlichkeitstheory. verw. Geb., (1977). **38**, 339-351.
- [12] IWANIEC T., P. KOSKELA P., AND ONNINEN J. *Mapping of finite distortion: Monotonicity and Continuity*. Invent. Math. 144 (2001), 507-531.
- [13] KLICNAROV'A JANA. *Central limit theorem for Hölder processes on R^m cube*. Comment.Math.Univ.Carolin. **48**, 1, (2007), 83-91.
- [14] KOZATCHENKO YU. V., OSTROVSKY E.I. (1985). *The Banach Spaces of random Variables of subgaussian type*. Theory of Probab. and Math. Stat. (in Russian). Kiev, KSU, **32**, 43-57.
- [15] LEDOUX M., TALAGRAND M. (1991) *Probability in Banach Spaces*. Springer, Berlin, MR 1102015.
- [16] LIFLYAND E., OSTROVSKY E., SIROTA L. *Structural Properties of Bilateral Grand Lebesgue Spaces*. Turk. J. Math.; **34** (2010), 207-219.
- [17] OSEKOWSKI A. *A Note on Burkholder-Rosenthal Inequality*. Bull. Polish Academy of Science, Math., 60, (2012), 177-185.
- [18] OSTROVSKY E.I. (1999). *Exponential estimations for random Fields and its applications (in Russian)*. Moscow - Obninsk, OINPE.
- [19] OSTROVSKY E. AND SIROTA L. *Moment Banach spaces: theory and applications*. HIAT Journal of Science and Engineering, **C**, Volume 4, Issues 1-2, pp. 233-262, (2007).

- [20] OSTROVSKY E. AND SIROTA L. *Module of continuity for the functions belonging to the Sobolev-Grand Lebesgue Spaces.* arXiv:1006.4177v1 [math.FA] 21 Jun 2010
- [21] OSTROVSKY E., SIROTA L. *Continuity of Functions belonging to the fractional Order Sobolev's-Grand Lebesgue Spaces.* arXiv:1301.0132v1 [math.FA] 1 Jan 2013
- [22] OSTROVSKY E., ROGOVER E. *Exact exponential Bounds for the random Field Maximum Distribution via the Majorizing Measures (Generic Chaining.)* arXiv:0802.0349v1 [math.PR] 4 Feb 2008
- [23] OSTROVSKY E.I. (2002). *Exact exponential Estimations for Random Field Maximum Distribution.* Theory Probab. Appl. 45 v.3, 281-286.
- [24] OSTROVSKY E., SIROTA L. *Monte-Carlo method for multiple parametric integrals calculation and solving of linear integral Fredholm equations of a second kind, with confidence regions in uniform norm.* arXiv:1101.5381v1 [math.FA] 27 Jan 2011
- [25] OSTROVSKY E., SIROTA L. *Moment Banach spaces: theory and applications.* HAIT Journal of Science and Engineering C, Volume 4, Issues 1-2, pp. 233-262, 2007. Holon Institute of Technology.
- [26] OSTROVSKY E., SIROTA L. *Schlömilch and Bell series for Bessel's functions, with probabilistic applications.* arXiv:0804.0089v1 [math.CV] 1 Apr 2008
- [27] OSTROVSKY E. *Bide-side exponential and moment inequalities for tail of distribution of polynomial martingales.* arXiv: math.Pr/0406532 V1 Jun 2004.
- [28] OSTROVSKY E., SIROTA L. *Moment and tail estimates for martingales and martingale transform, with application to the martingale limit theorem in Banach spaces.* arXiv:1206.4964v1 [math.PR] 21 Jun 2012
- [29] PROKHOROV YU.V. *Convergence of Random Processes and Limit Theorems of Probability Theory.* Probab. Theory Appl., (1956), V. 1, 177-238.
- [30] ROSENTHAL H.P. *On the subspaces of L_p ($p \geq 2$) spanned by sequences of independent Variables.* Israel J. Math., 1970, V.3 pp. 273-253.
- [31] HALL P., HEYDE C.C. *Martingale Limit Theory and Applications.* Academic Press, New York. (1980)
- [32] OSTROVSKY E., SIROTA L. *Sharp moment estimates for polynomial martingales.* arXiv:1410.0739v1 [math.PR] 3 Oct 2014
- [33] PIZIER G. *Les inegalites de Khintchine-Kahane dapres C. Borell: Sem. Geom: Espaces Banach, Ec. Polytech: Cent: Math: 1977-1978, Expose 7, (1-14).*
- [34] PIZIER G. *Conditions d'entropic assurant la continuite de certains processus et applications a l'analyse harmonique.* Seminaire d'analyse fonctionale, (1980). Exp. 13, p. 580-596.

- [35] RATCHKAUSKAS A., SUQUET CH. *Central limit theorems in Hölder topologies for Banach space valued random fields*. Teor. Veroyatnost. i Primenen., 2004, Volume 49, Issue 1, Pages 109-125, (in Russian).
- [36] RATCHKAUSKAS A., SUQUET CH. *Necessary and sufficient condition for the Hölderian functional central limit theorem*. J. Theoret. Probab. **17**, (2004), 221-243.
- [37] TALAGRAND M. (1990), Sample boundedness of stochastic processes under increment conditions. Annals of Probability 18, N. 1, 1-49, MR1043935.
- [38] TALAGRAND M. (1992). A simple proof of the majorizing measure theorem. Geom. Funct. Anal. 2, no. 1, 118-125. MR 1143666
- [39] WALK H. *A functional central limit theorem for martingales in $C(K)$ and its application to sequential estimates*. Journal für die reine und angewandte Mathematik, (Crelles Journal), 11 Oct 2014, Volume 1980, issue 314, 10-54.